DYNAMIC ECONOMETRIC MODELS
Vol. 3 – Nicholas Copernicus University – Toruń – 1998

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ARMA representation and state space representation of time series

1. Introduction

It may be noticed that the state space models gain more and more popularity[1] in econometric analysis. There is one reason of this focus, namely that the wide class of models (e.g. linear, non-linear, multiequational[2], stationary, nonstationary, time series with trend and seasonality[3]) may be written in the form of state space model. In polish literature some exampl of applying the state space and its methods may be found in e.g. optimal control theory[4], optimal filtration, determining different multipliers of econometric model[5].

The following questions arise: How do the state space methods stand in comparison with classical econometric models such as the ARMA models?

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[1] For example J.Jakubczyc (1996), K.Strza³a (1994) and also J.B.Gajda (1993).

[2] The multi-equational form of state space model may be found in A.Michalczewskiej-Litwy (1977).

[3] The mentioned and others applications of state space may be found amongst others in L.Fahrmeir, G.Tutz (1996).

[4] The application of state space in the optimal control of non-linear models is presented in K. Strza³y (1994) and also J.B.Gajdy (1993).

[5] The use of state space method to determine multipliers is contained in A.Sikorskiego (1980).

Which relations are observed between the ARMA model and the state space model?

The purpose of this paper is to approach the notion of state space and state space representation of stochastic process, and also to present the relationship between the state space models and the ARMA models.

2. State space

The *state* is a minimum number of independent date characterizing entirely the „position" of the dynamic system in a given moment, but also having the ability to predict the behavior of system in the future. Let us assume the initial state (in t_0 moment) is known. In a moment t the state may be written as a function of time

$$x(t) = \left[x_1(t), x_2(t), \ldots, x_n(t)\right]^T . \quad (1)$$

The vector $x(t)$ is called „state vector" and variables $x_1(t), x_2(t), \ldots$ - state variables. The space determined by the state vectors $x(t)$, is called the state space[6].The dimension of state space is equal to the number of state variables.

The state space model is described by the state equation (2) and the output equation (observation equation) (3).

$$x_{t+1} = F_t \cdot x_t + D_t \cdot u_t + G_t \cdot w_t \quad (2)$$

$$z_t = H_t \cdot x_t + v_t \quad (3)$$

where

- x_t - n-dimensional state vector,
- z_t - m-dimensional output (observations) vector,
- u_t, - r-dimensional control vector,
- v_t, - m-dimensional error measure vector (white noise),
- w_t - p-dimensional disturbance vector (white noise),
- F_t - the $n \times n$ state matrix, it presents relationships among respective state variables,
- D_t - the $n \times r$ control matrix, it shows the way of influencing state variables by constrains,

[6] The more detailed description of the state and the state space may be found in: K.Ogata (1974), A.Michalczewska-Litwa (1977), B.W¹sik, A.Litwa, J.Skrzypek, (1986), J.Gutenbaum, (1975), S.Grzesiak (1995).

- G_t - the *n x p* disturbance matrix, it represents the way of shifting disturbances on respective elements,
- H_t - the *m x n* output matrix; it shows how state variables are transformed into output variables.

The model described by equations (2) and (3) is presented at figure 1.

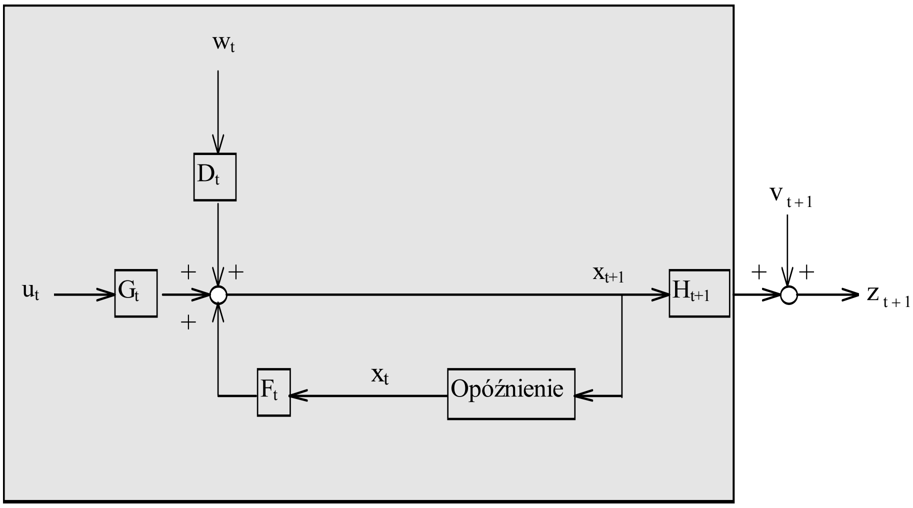

Fig. 1. Block scheme of state space

The block scheme shows not only the chief point of state space, but also the essence of economic phenomena at all. What we are able to observe (measure, weigh, ...) it is merely the final effect of many processes. The input equation displays the way of influencing our observations by state variables. The changes going through inside a process are described by the state equation. The variables appearing in an econometric model which are lagged by more then one period are treated as new state variables[7].

In economic applications the state vector is often identified with the output vector (although the state vector may be different from the output vector).
In connection with it the matrix H often takes the form of the unit matrix or vector $\begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}$. The state space representation for a linear stochastic process z_t has the form:

[7] This approach can be found in: K.Ogata (1974), A.Michalczewska-Litwa (1977).

$$\begin{cases} x_{t+1} = F \cdot x_t + G \cdot \varepsilon_{t+1} \\ z_t = H \cdot x_t \end{cases} . \tag{4}$$

3. The ARMA model

The ARMA model does not require detailed presentation as it is wide known[8]. The ARMA representation of stationary stochastic process has the form:

$$\phi(B) z_t = \theta(B)\varepsilon_t \tag{5}$$

or

$$z_t = \phi_1 z_{t-1} + \ldots + \phi_p z_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \ldots - \theta_q \varepsilon_{t-q} \tag{6}$$

$$z_{t+p} = \phi_1 z_{t+p-1} + \ldots + \phi_p z_t + \varepsilon_{t+p} - \theta_1 \varepsilon_{t+p-1} - \ldots - \theta_q \varepsilon_{t+p-q} \tag{6'}$$

where

$\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$ is an autoregressive polynomial of p-order,

$\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$ is a moving average operator of q-order,

ε_t - white noise.

The process (5) may be written in the form:

$$z_t = \phi^{-1}(B)\theta(B)\varepsilon_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j} \tag{7}$$

where $\psi_0 = 1$.

4. The relationship between state space and ARMA models

In subsection 1^0 the state space representation will be found for process z_t having the ARMA representation[9]. Whereas in subsection 2^0 the ARMA representation will be derived for the process z_t described by the state equation and output equation[10].

1^0 Let us consider the stationary ARMA(p,q) model described by equation (5)-(7).

[8] The description of ARMA model is presented in: G.E.P.Box, G.M.Jenkins (1983) and also L.Talaga, Z.Zieliñski (1986).

[9] This transformation by the means of Kalman filter is also presented in S.Grzesiak (1995).

[10] This relationship as well the previous one between the ARMA representation and state space representation may be found in W.Wei (1990).

$$z_{t+i} = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t+i-j} \tag{8}$$

Let

$$z_{t+i,t} = E\left[z_{t+i} \middle| z_k, k \le t\right] = \sum_{j=i}^{\infty} \psi_j \varepsilon_{t+i-j} \,. \tag{9}$$

Thereby

$$\begin{aligned} z_{t+i,t+1} &= E\left[z_{t+i} \middle| z_k, k \le t+1\right] = \sum_{j=i-1}^{\infty} \psi_j \varepsilon_{t+i-j} = \\ &= \psi_{i-1}\varepsilon_{t+1} + \sum_{j=i}^{\infty} \psi_j \varepsilon_{t+i-j} = \psi_{i-1}\varepsilon_{t+1} + z_{t+i,t} \end{aligned} \,. \tag{10}$$

Hence

$$\begin{aligned} z_{t+1,t+1} &= z_{t+1,t} + \varepsilon_{t+1} \\ z_{t+2,t+1} &= z_{t+2,t} + \psi_1 \varepsilon_{t+1} \\ &\dots \\ z_{t+p-1,t+1} &= z_{t+p-1,t} + \psi_{p-2} \varepsilon_{t+1} \\ z_{t+p,t+1} &= z_{t+p,t} + \psi_{p-1} \varepsilon_{t+1} \end{aligned} \tag{11}$$

Calculating the conditional expectation value for a process of the form (6’) we obtain

$$z_{t+p,t} = \phi_1 z_{t+p-1,t} + \ldots + \phi_p z_{t,t} \,. \tag{12}$$

Thereby the last equation in (11) in consideration of (12) has the form

$$z_{t+p,t+1} = \phi_p z_{t,t} + \phi_{p-1} z_{t+1,t} + \ldots + \phi_1 z_{t+p-1,t} + \psi_{p-1}\varepsilon_{t+1} \tag{13}$$

For $i \ge 0$ $z_{t+p+i,t}$ is a function of $z_{t,t}, z_{t+1,t}, \ldots, z_{t+p-1,t}$.

$$z_{t+p+1,t} = \phi_1 z_{t+p,t} + \ldots + \phi_p z_{t+1,t} = f\left(z_{t,t}, z_{t+1,t}, \ldots, z_{t+p-1,t}\right) \tag{14}$$

Hence the vector $\left[z_{t,t}, z_{t+1,t}, \ldots, z_{t+p-1,t}\right]$ is a state space vector, and the state space representation of ARMA model is given by (15) and (16).

$$\begin{bmatrix} z_{t+1,t+1} \\ z_{t+2,t+1} \\ \dots \\ z_{t+p,t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \phi_p & \phi_{p-1} & \phi_{p-2} & \phi_{p-3} & \dots & \phi_1 \end{bmatrix} \cdot \begin{bmatrix} z_{t,t} \\ z_{t+1,t} \\ \dots \\ z_{t+p-1,t} \end{bmatrix} + \begin{bmatrix} 1 \\ \psi_1 \\ \dots \\ \psi_{p-1} \end{bmatrix} \cdot \varepsilon_{t+1} \quad (15)$$

$$z_t = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \cdot \begin{bmatrix} z_{t,t} \\ z_{t+1,t} \\ \dots \\ z_{t+p-1,t} \end{bmatrix} \quad (16)$$

2⁰ Let us assume that the process z_t has the state space representation in the form (4). The characteristic polynomial of matrix F is given by (17)

$$|\lambda I - F| = \sum_{i=0}^{p} \phi_i \lambda^{p-i} , \quad (17)$$

where $\phi_0 = 1$. From the Cayley-Hamilton statement is known that

$$\sum_{i=0}^{p} \phi_i F^{p-i} = 0 . \quad (18)$$

Carring out sequential substitution into first equation (4) we hawe

$$\begin{aligned} x_{t+i} &= F \cdot x_{t+i-1} + G \cdot \varepsilon_{t+i} = \\ &= F\left(F \cdot x_{t+i-2} + G \cdot \varepsilon_{t+i-1}\right) + G \cdot \varepsilon_{t+i} = \\ &= F^2 x_{t+i-2} + F \cdot G \cdot \varepsilon_{t+i-1} + G \cdot \varepsilon_{t+i} = \\ &\dots\dots \\ &= F^i x_t + F^{i-1} \cdot G \cdot \varepsilon_{t+1} + \dots + G \cdot \varepsilon_{t+i} \end{aligned} \quad (19)$$

Multiplying the second equation in (4) by respective ϕ_i we obtain

$$\begin{aligned} z_{t+p} &= H \cdot x_{t+p} = H\left(F^p x_t + F^{p-1} \cdot G \cdot \varepsilon_{t+1} + \dots + G \cdot \varepsilon_{t+p}\right) \\ \phi_1 z_{t+p-1} &= H \cdot \phi_1 x_{t+p-1} = H \cdot \phi_1 \left(F^{p-1} x_t + F^{p-2} \cdot G \cdot \varepsilon_{t+1} + \dots + G \cdot \varepsilon_{t+p-1}\right) \\ &\dots\dots\dots\dots \\ \phi_{p-1} z_{t+1} &= H \cdot \phi_{p-1} x_{t+1} = H \cdot \phi_{p-1}\left(F x_t + G \cdot \varepsilon_{t+1}\right) \\ \phi_p z_t &= H \cdot \phi_p x_t \end{aligned} \quad (20)$$

Summing up right and left sides of (20) we have

$$z_{t+p}+\phi_1 z_{t+p-1}+\ldots+\phi_{p-1}z_{t+1}+\phi_p z_t = H\left(F^p x_t + F^{p-1}\cdot G\cdot\varepsilon_{t+1}+\ldots+G\cdot\varepsilon_{t+p}\right)+$$
$$+H\cdot\phi_1\left(F^{p-1}x_t+F^{p-2}\cdot G\cdot\varepsilon_{t+1}+\ldots+G\cdot\varepsilon_{t+p-1}\right)+\ldots$$
$$\ldots+H\cdot\phi_{p-1}\left(Fx_t+G\cdot\varepsilon_{t+1}\right)+H\cdot\phi_p x_t =$$
$$= H\left(F^p+\phi_1 F^{p-1}+\ldots+\phi_{p-1}F+\phi_p I\right)x_t +$$
$$+H\left(F^{p-1}+\phi_1 F^{p-2}+\ldots+\phi_{p-1}I\right)G\cdot\varepsilon_{t+1}+..+HG\varepsilon_{t+p}$$

Let

$$\theta_i = H\left(F^i+\phi_1 F^{i-1}+\ldots+\phi_i I\right)G, \qquad i=0,1,\ldots,p-1. \tag{21}$$

Then considering (18) the process z_t has the ARMA representation in the form

$$z_{t+p}+\phi_1 z_{t+p-1}+\ldots+\phi_{p-1}z_{t+1}+\phi_p z_t = \theta_0\varepsilon_{t+p}+\theta_1\varepsilon_{t+p-1}+\ldots+\theta_{p-1}\varepsilon_{t+1} \tag{22}$$

or by equivalent equation

$$z_t+\phi_1 z_{t-1}+\ldots+\phi_{p-1}z_{t-(p-1)}+\phi_p z_{t-p} = \theta_0\varepsilon_t+\theta_1\varepsilon_{t-1}+\ldots+\theta_p\varepsilon_{t-(p-1)}. \tag{22'}$$

To obtain the representation (22) the Cayley-Hamilton statement was used and as a result the ARMA coefficients were obtained.

5. An example

1⁰ Let us take the ARMA(2,1) model described by equation

$$z_t = \phi_1 z_{t-1}+\phi_2 z_{t-2}+\varepsilon_t-\theta_1\varepsilon_{t-1}, \tag{23}$$

or by equivalent equation

$$z_{t+2} = \phi_1 z_{t+1}+\phi_2 z_t+\varepsilon_{t+2}-\theta_1\varepsilon_{t+1}. \tag{23'}$$

Considering that in the equation (23) is only one variable lagged by more that some period then the state vector is two-dimensional.

Let us denote

$$z_{t+1,t+1}=z_{t+1}, \quad z_{t+2,t+1}=z_{t+2} \quad \Rightarrow \quad x_{t+1}=\begin{bmatrix} z_{t+1,t+1} \\ z_{t+2,t+1}\end{bmatrix}$$

Equation (23) may be written as moving average

$$z_t = \left(1-\phi_1 B-\phi_2 B^2\right)^{-1}\left(1-\theta_1 B\right)\varepsilon_t = \sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j} \tag{24}$$

where $\psi_0 = 1$, $\psi_1 = \phi_1-\theta_1,\ldots$. Substituting it to (15) and (16) we have matrix F, G and H:

$$F = \begin{bmatrix} 0 & 1 \\ \phi_2 & \phi_1 \end{bmatrix}, \; G = \begin{bmatrix} 1 \\ \psi_1 \end{bmatrix}, \; H = [1,0]. \tag{25}$$

The state space representation of model (23) has the following form:

$$\begin{bmatrix} z_{t+1,t+1} \\ z_{t+2,t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \phi_1 & \phi_2 \end{bmatrix} \cdot \begin{bmatrix} z_{t,t} \\ z_{t+1,t} \end{bmatrix} + \begin{bmatrix} 1 \\ \psi_1 \end{bmatrix} \varepsilon_{t+1} \tag{26}$$

$$z_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} z_{t,t} \\ z_{t+1,t} \end{bmatrix} \tag{27}$$

Equation (27) may be omitted because it contributes nothing to the analysis.

2⁰ Let us assume that the process z_t has the state space representation in the form

$$\begin{cases} x_{t+1} = F \cdot x_t + G \cdot \varepsilon_{t+1} \\ z_t = H \cdot x_t \end{cases}. \tag{28}$$

where matrix *F, G, H* have the *2x2, 2x1* and *1x2* dimensional respectively.

Now the characteristic polynomial of matrix *F*.

$$F^2 + \phi_1 F + \phi_2 I = 0 \tag{29}$$

From the state equation (28) results that

$$x_{t+2} = F \cdot x_{t+1} + G \cdot \varepsilon_{t+2}. \tag{30}$$

Substituting equation (30) into the state equation (28) we obtain.

$$\begin{aligned} x_{t+2} &= F \cdot \left(F \cdot x_t + G \cdot \varepsilon_{t+1} \right) + G \cdot \varepsilon_{t+2} = \\ &= F^2 x_t + FG\varepsilon_{t+1} + G \cdot \varepsilon_{t+2} \end{aligned} \tag{31}$$

Multiplying the output equation (28) by respective coefficients of characteristic polynomial (29) we have

$$z_{t+2} = H \cdot x_{t+2} = H\left(F^2 x_t + FG\varepsilon_{t+1} + G\varepsilon_{t+2} \right), \tag{32}$$

$$\phi_1 \cdot z_{t+1} = \phi_1 \cdot H \cdot x_{t+1} = \phi_1 H\left(Fx_t + G \cdot \varepsilon_{t+1} \right),$$

(33)

$$\phi_2 \cdot z_t = \phi_p \cdot H \cdot x_t. \tag{34}$$

As a result of summing up members of equation (32)-(34) we obtain

$$
\begin{aligned}
z_{t+2} + \phi_1 z_{t+1} + \phi_2 z_t &= H\left(F^2 x_t + FG\varepsilon_{t+1} + G\varepsilon_{t+2}\right) + \\
&\quad + \phi_1 H\left(Fx_t + G\varepsilon_{t+1}\right) + \phi_2 Hx_t = \\
&= H\left(F^2 + \phi_1 F + \phi_2 I\right)x_t + \\
&\quad + H\left(F + \phi_1 I\right)G\varepsilon_{t+1} + HG\varepsilon_{t+2} = \\
&= H\left(F + \phi_1 I\right)G\varepsilon_{t+1} + HG\varepsilon_{t+2}
\end{aligned} \quad . \quad (35)
$$

Denoting

$$\theta_0 = HG, \quad \theta_1 = H\left(F + \phi_1 I\right)G \quad (36)$$

we have

$$z_{t+2} + \phi_1 z_{t+1} + \phi_2 z_t = \theta_0 \varepsilon_{t+2} + \theta_1 \varepsilon_{t+1}, \quad (37)$$

or equivalent form

$$z_t + \phi_1 z_{t-1} + \phi_2 z_{t-2} = \theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1}. \quad (37')$$

The process z_t described by equation (37) or (37') has the ARMA(2,1) representation.

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