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Predictive properties of the autoregressive and state space models – a comparison

The paper is aimed to compare estimation results of the models generated by ARMA process with the respective state space models. Both modelling representations are also investigated with respect to their predictive properties.

1. The ARMA model

The ARMA representation[1] of stationary stochastic process z_t has the form:

$$f(B)z_t = q(B)e_t \tag{1}$$

or

$$z_t = f_1 z_{t-1} + ... + f_p z_{t-p} + e_t - q_1 e_{t-1} - ... - q_q e_{t-q} \tag{2}$$

where:

$f(B) = 1 - f_1 B - ... - f_p B^p$ is an autoregressive polynomial of *p*-order,

$q(B) = 1 - q_1 B - ... - q_q B^q$ is a moving average operator of *q*-order,

e_t is white noise.

The process (1) may be written in the form:

[1] The description of ARMA model is presented in: Box, Jenkins (1983) and also Talaga, Zieliński (1986).

$$z_t = f^{-1}(B)q(B)e_t = \sum_{j=0}^{\infty} y_j e_{t-j} \tag{3}$$

where:

$y_0 = 1$.

2. The state space model

The state equation and the output equation (observation equation) formulate the state space model[2]. The state space representation (SS hereafter) for stationary process z_t has the form:

$$\mathbf{x}_{t+1} = \mathbf{F}\mathbf{x}_t + \mathbf{G}v_t \tag{4}$$

$$z_t = \mathbf{H}\mathbf{x}_t + v_t \tag{5}$$

where:

$\operatorname{cov} v_t = \Delta$,

$\mathbf{F}_{n\times n}$, $\mathbf{G}_{n\times p}$, $\mathbf{H}_{l\times n}$ are respective state matrix disturbances matrix and output matrix.

3. Equivalence of the ARMA and space state models

The ARMA (p, q) model (described by formulas (1)–(3)) may be written as the state space model.

Let $m = \max(p,q)$ and $f_j = 0$ dla $j > p$. Then the state space model (4)–(5) takes the form:

$$x_{t+1} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ f_m & f_{m-1} & f_{m-2} & f_{m-3} & \dots & f_1 \end{bmatrix} \cdot x_t + \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_m \end{bmatrix} \cdot e_{t+1} \tag{6}$$

[2] The state space model was analysed among others in: Aoki (1990), Ogata (1974), Michalczewska-Litwa (1977), Wąsik, Litwa, Skrzypek (1986), Gutenbaum (1975), Grzesiak (1995) and Górka (1997).

$$z_t = [1 \quad 0 \quad \ldots \quad 0] \cdot x_t \qquad t = 0, \pm 1, \ldots \tag{7}$$

On the contrary, the space state model, given in (6)–(7) may be written as the ARMA model

$$z_{t+p} + f_1 z_{t+p-1} + \ldots + f_{p-1} z_{t+1} + f_p z_t = q_0 e_{t+p} + q_1 e_{t+p-1} + \ldots + q_{p-1} e_{t+1}$$

where:
$f_1, \ldots, f_p$ are coefficients of characteristic polynomial of matrix **F**:

$$\left| l\mathbf{I} - \mathbf{F} \right| = \sum_{i=0}^{p} f_i \, l^{p-i} \tag{8}$$

and $q_i = \mathbf{H}\left(\mathbf{F}^i + f_1 \mathbf{F}^{i-1} + \ldots + f_i \mathbf{I}\right)\mathbf{G}, \;\; i = 0, 1, \ldots, p-1$.

The relationship between minimal orders of ARMA representation and minimal dimension of the state space representation is strict. Let $\underline{p}, \underline{q}$ be a respective minimal order of the autoregressive polynomial and minimal order of the moving average operator. Let $\underline{K}$ be minimal dimension of state space model. Then[3]

$$\underline{K} = \max\left(\underline{p}, \underline{q}\right). \tag{9}$$

4. The state space model estimation

The Henkel matrix is given for observation z_t

$$\mathsf{H}_{J,K} = \begin{bmatrix} \Lambda_1 & \Lambda_2 & & \Lambda_K \\ \Lambda_2 & \Lambda_3 & & \Lambda_{K+1} \\ & & & \\ \Lambda_J & \Lambda_{J+1} & & \Lambda_{J+K-1} \end{bmatrix}. \tag{10}$$

[3] The proof of (9) can be found in Gourieroux., Monfort (1997).

The Hankel matrix is the covariance matrix between the future stacked z_t^+ and the stacked data z_{t-1}^-. The estimate of the matrix (10) elements is expressed by the formula (11)

$$\hat{\Lambda}_k = \frac{1}{N} z_t^T z_{t+k} . \tag{11}$$

Let the singular value decomposition of the matrix H be given by $\mathsf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$, where $\boldsymbol{\Sigma}$ is a diagonal matrix and **U**, **V** are orthogonal matrices.

For each matrix of the SS model (4)–(5) the following estimates are given[4]:

$$\hat{\mathbf{F}} = \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{U}^T \mathsf{H}^A \mathbf{V} \boldsymbol{\Sigma}^{-\frac{1}{2}} \tag{12}$$

$$\hat{\mathbf{M}} = \boldsymbol{\Sigma}^{-\frac{1}{2}} \mathbf{U}^T \mathsf{H}_{\bullet,1} \tag{13}$$

$$\hat{\mathbf{H}} = \mathsf{H}_{1,\bullet} \mathbf{V} \boldsymbol{\Sigma}^{-\frac{1}{2}} \tag{14}$$

where:

$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^{\frac{1}{2}} \boldsymbol{\Sigma}^{\frac{1}{2}}$

$\boldsymbol{\Sigma}^{-\frac{1}{2}}$ – the Moore-Penrose pseudoinverse,

$\mathsf{H}_{\bullet,1}$, $\mathsf{H}_{1,\bullet}$ are respective first column and top row of matrix H,

H^A – the covariance matrix[5] between z_{t+1}^+ and z_{t-1}^-,

$\mathbf{M} = \mathbf{F}\boldsymbol{\Pi}\mathbf{H}^T + \mathbf{G}\Delta$, where $\boldsymbol{\Pi} = E\left(\mathbf{x}_t, \mathbf{x}_t^T\right)$.

5. The estimation and prediction results analysis

As an example of the ARMA representation the AR(2) model was chosen to show some properties of the representations in interest. The AR(2) series were generated 500 times in Statistica 5.1 package. The observations number was 300 for each series. To save space the autoregressive coefficient f_1 was stable over all series and equal 0.5, while f_2 coefficient changed by 0.1. Only stationary combinations of coefficients were taken into account. The white noise process was generated as normally distributed N(0,1). The generated series were estimated to check the resulted AR(2) models and their state space equivalents.

[4] The estimation formulas derivation is avalaible in Aoki (1990).
[5] Hankel matrix with additional row.

The SS matrices were estimated using the formulas (11)–(14), with the programme written in Statistica Basic, available in Statistica package. From (12) is clear, that Hankel matrix is of the same dimension as the state matrix. The state matrix is a quadratic one, then $J = K$. The minimum dimension of the SS model and the AR order are related according to equation (9). Thus

$$J = K = \max(p, q) = \max(2{,}0) = 2 .$$

The autoregressive coefficients were estimated independent by OLS and by way of state space. The ratios of the models which coefficients' estimates, by way of the SS and AR, do not exceed 10%, 20% and 30% of the respective values set in the modulus are collected in table 1.

Table 1. The ratios of the models which coefficients' estimates do not exceed 10%, 20% and 30% of the respective values set.

f_2	$\lvert\Delta f\rvert < 0{,}1f$		$\lvert\Delta f\rvert < 0{,}2f$		$\lvert\Delta f\rvert < 0{,}3f$	
	f_1	f_2	f_1	f_2	f_1	f_2
0,4	15,4%	12,6%	31,4%	27,4%	44,6%	37%
0,3	8,8%	5,8%	18%	12,6%	27%	18,6%
0,2	5,4%	2,8%	11,4%	5%	17,8%	9,2%
0,1	3,4%	0,8%	7,4%	2%	11%	2,8%
-0,1	2,4%	1%	6,2%	2,6%	10,4%	3,6%
-0,2	6,8%	17,6%	14%	30%	19,4%	41,8%
-0,3	12,2%	23,8%	25,2%	47,2%	38,2%	67%
-0,4	23,6%	29%	47,6%	59,2%	64%	81,2%
-0,5	34%	48,4%	59,4%	84%	79,4%	95,2%
-0,6	52,6%	69,6%	84%	97,4%	97%	99,4%
-0,7	53,6%	71,2%	87,8%	98,8%	98,4%	100%
-0,8	75,8%	81,8%	98,4%	99,6%	99,6%	100%
-0,9	92,2%	99,8%	100%	100%	100%	100%

* Computed by the author.

The relationship between values of coefficients arbitrarily set and modulus of the distance of the parameters' estimates, using the two independent methods, can be noticed. The less is the value f_2, the less the distances in estimates, not only of the mentioned parameter. Besides, the ratio of the models better fitted is different for the same modulus of the parameter f_2.

The projection for 5 periods forward, for each series, was done. The forecasts were calculated independently for the SS and AR models. It can be seen that for the SS model, for some coefficient f_2 values, the second value of the forecast, i.e. the 302 fitted value is the local extreme. Let us assume that the local maximum corresponds to the situation that 302 value of the projection is bigger than the observation in time 300. The local minimum implies that the projection value in time *T*+2 is less then in the *T*-s observation. In the case of the AR models the described relationship cannot be confirmed.

It was assumed, for the AR model, that if the first three forecast values constitute increasing sequence of numbers, then the projection value in 302 is greater than the observation value in 300. If the opposite situation takes place, the forecast values take form of the decreasing sequence of numbers and value in 300 is bigger than in 302. The ratios of the proper forecasts for the models under study, as well as the corresponding number of series for given f_2, in which the SS projection possesses the local extreme are presented in table 2.

Table 2. The ratios of the correct forecasts for models investigated.

f_2	The SS model's correct forecast ratio		The AR model's correct forecast ratio		min, max in PS	The ratio of min, max in good forecast
	number	%	number	%	number	%
0,4	0	0	121	24,2	0	0
0,3	0	0	203	40,6	0	0
0,2	0	0	283	56,6	0	0
0,1	1	0,2	318	63,6	3	33,3
-0,1	82	16,4	344	68,8	119	68,9
-0,2	269	53,8	307	61,4	376	71,5
-0,3	374	74,8	270	54	491	76,2
-0,4	373	74,6	237	47,4	499	74,7
-0,5	401	80,2	210	42	500	80,2
-0,6	406	81,2	218	43,6	500	81,2
-0,7	417	83	187	37,4	500	83
-0,8	439	87,8	175	35	500	87,8
-0,9	445	89	180	36	500	89

* Computed by the author.

For $f_2 \leq -0{,}4$ each forecast sequence calculated on the basis of the SS model possesses the local extreme. The ratio of cases when the forecast was correct and the local extreme took place, independently of the value of f_2 is bigger than 74%. This means that if the local extremes appear then in at least 74 cases for each 100 we are able to forecast the direction of the series changes correctly. For the AR (2) model the right forecasts were obtained in (24.2%;

68.8%) cases. The last but not least remark is the following: the less is the distance between model parameters' values the forecasts are more effective, i.e. we are able to indicate the direction of the changes more properly.

References

Aoki M., (1990), *State Space Modeling of Time Series*, Springer-Verlang, Berlin.

Box G.E.P., Jenkins G.M., (1983), *Analiza szeregów czasowych*, PWN, Warszawa.

Brockwell P.J., Davis R.A., (1996), *Introduction to Time Series and Forecasting*, Springer.

Brockwell P.J., Davis R.A., (1991), *Time Series: Theory and Methods*, Springer.

Gourieroux Ch., Monfort A., (1997), *Time Series and Dynamic Models.*Cambridge.

Górka J., (1997), „Reprezentacja ARMA i reprezentacja przestrzeni stanów szeregów czasowych", w *Dynamiczne Modele Ekonometryczne,* UMK, Toruń.

Grzesiak S., (1995), „Struktura stochastycznie zmiennych modeli parametru a równanie filtru Kalmana", w *Dynamiczne Modele Ekonometryczne,* Toruń.

Gutenbaum J., (1975), *Problem teorii regulatorów*, WNT, Warszawa.

Kiełbasiński A., Schwetlick H., (1992), *Numeryczna algebra liniowa*, WNT, Warszawa.

Michalczewska-Litwa A., (1977), „Postać wielorównaniowego modelu ekonometrycznego w przestrzeni stanów", *Przegląd statystyczny*, zeszyt 1, str.41–54.

Ogata K., (1974), *Metody przestrzeni stanów w teorii sterowania*, WNT, Warszawa.

Talaga L., Zieliński Z., (1986), *Analiza spektralna w modelowaniu ekonometrycznym*, PWN, Warszawa.

Wei W. W. S., (1990), *Time series analysis*, Addison–Wesley Publishing Company, Inc, Berlin.